

# ON THE OCCURRENCE OF LARGE POSITIVE HECKE EIGENVALUES FOR $GL(2)$

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**ABSTRACT.** Let  $\pi$  be a self-dual cuspidal automorphic representation for  $GL(2)/\mathbb{Q}$ . We show that there exists a positive upper Dirichlet density of primes at which the associated Hecke eigenvalues of  $\pi$  are larger than a specified positive constant.

## 1. INTRODUCTION

Let  $\pi$  be a cuspidal automorphic representation for  $GL(2)/\mathbb{Q}$  that is self-dual. To each prime  $p$  at which  $\pi$  is not ramified is associated a Hecke eigenvalue, denoted by  $a_p = a_p(\pi)$ . The values taken by sequences  $\{a_p(\pi)\}_p$  of Hecke eigenvalues have been long-studied from various points of view. In 1994, J.-P. Serre asked (see appendix of [7]) whether it is possible to find positive constants  $c, c'$  such that, for all  $\epsilon > 0$ , there exist infinitely many  $a_p$  greater than  $c - \epsilon$  and infinitely many  $a_p$  less than  $-c' + \epsilon$ . He then proved such results for the case of (certain) modular forms, and asked if similar results can be shown to hold in the case of Maass forms.

In [8] we proved, for any self-dual cuspidal automorphic representation  $\pi$  for  $GL(2)/\mathbb{Q}$ , that for any positive  $\epsilon$  there exist infinitely many primes  $p$  such that

$$a_p > 0.905\dots - \epsilon$$

and if  $\pi$  is non-dihedral, then there exist infinitely many primes  $p$  such that

$$a_p < -1.164\dots + \epsilon$$

(precise expressions for the constants are available in [8]). Note that a cuspidal automorphic representation  $\pi$  for  $GL(2)$  is said to be *dihedral* if it is associated to a 2-dimensional irreducible Artin representation  $\rho$  that is of dihedral type, meaning that the image of  $\rho$  in  $PGL_2(\mathbb{C})$  is isomorphic to a dihedral group. Furthermore,  $\pi$  is said to be of *solvable polyhedral type* if it is associated to a 2-dimensional irreducible Artin representation  $\rho$  that is of dihedral, tetrahedral, or octahedral type (which means that the projective image of  $\rho$  in  $PGL_2(\mathbb{C})$  is isomorphic to a dihedral group,  $A_4$ , or  $S_4$ , respectively).

A related question is to ask whether it is possible to obtain similar statements for not just an infinitude of primes but a positive upper Dirichlet density of primes. Recall that the *upper Dirichlet density* of a set  $S$  of primes is defined to be

$$\bar{\delta}(S) := \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in S} p^{-s}}{\log(1/(s-1))}.$$

The results of [8] relied on determining lower bounds on the asymptotic growth of certain Dirichlet series, but, because of the lack of knowledge of the Ramanujan

conjecture (or indeed the non-existence of any known uniform bound on the Satake parameters), these will not directly lead to a positive density result.

In this paper, we outline a method to circumvent this issue and obtain positive upper Dirichlet density results. By way of example, we have

**Theorem 1.1.** *Let  $\pi$  be a self-dual cuspidal automorphic representation for  $\mathrm{GL}(2)$  over  $\mathbb{Q}$  that is not of solvable polyhedral type. Then for any  $\epsilon > 0$ , the set*

$$\{p \mid a_p(\pi) > 0.778\dots - \epsilon\}.$$

*has an upper Dirichlet density of at least  $1/100$ .*

The exact value of the constant in the set condition is  $0.36729^{1/4}$ , which is determined in Section 3. The method we use would also allow a change in this constant to a smaller value so as to obtain a mild increase in the lower bound of the density. The proof relies in part on the deep work of Gelbart–Jacquet [1], Kim–Shahidi [4, 5], and Kim [3] on the automorphy of symmetric power lifts. In the next section, we will outline the ingredients used in the proof, and in Section 3 we will prove the theorem.

## 2. BACKGROUND

The proof will rely on the study of the asymptotic behaviour of various Dirichlet series, which we will briefly outline here and refer the reader to [8] for a more detailed explanation.

Given a cuspidal automorphic representation  $\pi$  for  $\mathrm{GL}(2)/\mathbb{Q}$ , let  $T$  be the (finite) set consisting of the archimedean place and the finite places at which  $\pi$  is ramified. Then the incomplete  $L$ -function (with respect to  $T$ ) that is associated to  $\pi$  can be defined in a right-half plane via an Euler product:

$$L^T(s, \pi) = \prod_{p \notin T} \det(I_2 - A_p(\pi)p^{-s})^{-1},$$

where  $I_2$  is the  $2 \times 2$  identity matrix and  $A_p(\pi) = \mathrm{diag}(\alpha_p(\pi), \beta_p(\pi)) \in \mathrm{GL}_2(\mathbb{C})$  is the matrix of Satake parameters associated to  $\pi$  at  $p$ . For any two cuspidal automorphic representations  $\pi_1$  for  $\mathrm{GL}(n)/\mathbb{Q}$  and  $\pi_2$  for  $\mathrm{GL}(m)/\mathbb{Q}$ , one can define (again in a suitable right-half plane) their incomplete Rankin–Selberg  $L$ -function:

$$L^T(s, \pi_1 \times \pi_2) = \prod_{p \notin T} \det(I_{nm} - A_p(\pi_1) \otimes A_p(\pi_2)p^{-s})^{-1}.$$

This  $L$ -function converges absolutely for  $\mathrm{Re}(s) > 1$ . At  $s = 1$  it has a simple pole iff  $\pi_1$  is dual to  $\pi_2$  [2], otherwise the  $L$ -function is invertible at that point [6].

In general one can define, for any positive integer  $k \leq 8$ , an incomplete  $k$ th product  $L$ -function as follows:

$$L^T(s, \pi^{\times k}) = \prod_{p \notin T} \det(I_{2^k} - A_p(\pi)^{\otimes k} p^{-s})^{-1}.$$

One can also define the following incomplete symmetric power  $L$ -functions:

$$\begin{aligned} L^T(s, \text{Sym}^2 \pi) &= \prod_{p \notin T} \det \left( I_3 - \begin{pmatrix} \alpha_p^2 & & \\ & \alpha_p \beta_p & \\ & & \beta_p^2 \end{pmatrix} p^{-s} \right)^{-1}, \\ L^T(s, \text{Sym}^3 \pi) &= \prod_{p \notin T} \det \left( I_4 - \begin{pmatrix} \alpha_p^3 & & & \\ & \alpha_p^2 \beta_p & & \\ & & \alpha_p \beta_p^2 & \\ & & & \beta_p^3 \end{pmatrix} p^{-s} \right)^{-1}, \\ L^T(s, \text{Sym}^4 \pi) &= \prod_{p \notin T} \det \left( I_5 - \begin{pmatrix} \alpha_p^4 & & & & \\ & \alpha_p^3 \beta_p & & & \\ & & \alpha_p^2 \beta_p^2 & & \\ & & & \alpha_p \beta_p^3 & \\ & & & & \beta_p^4 \end{pmatrix} p^{-s} \right)^{-1}. \end{aligned}$$

For the  $k$ th product  $L$ -functions, where  $k = 3, 4, 6$ , and  $8$ , we have the following  $L$ -function identities, using Clebsch–Gordon decompositions:

$$\begin{aligned} L^T(s, \pi^{\times 3}) &= L^T(s, \text{Sym}^3 \pi) L^T(s, \pi \otimes \omega)^2, \\ L^T(s, \pi^{\times 4}) &= L^T(s, \text{Sym}^4 \pi) L^T(s, \text{Sym}^2 \pi \otimes \omega)^3 L^T(s, \omega^2)^2, \\ L^T(s, \pi^{\times 6}) &= L^T(s, \text{Sym}^3 \pi \times \text{Sym}^3 \pi) L^T(s, \text{Sym}^3 \pi \times \pi \otimes \omega)^4 L^T(s, \pi \times \pi \otimes \omega^2)^4, \\ L^T(s, \pi^{\times 8}) &= L^T(s, \text{Sym}^4 \pi \times \text{Sym}^4 \pi) L^T(s, \text{Sym}^4 \pi \times \text{Sym}^2 \pi \otimes \omega)^6 \\ &\quad \cdot L^T(s, \text{Sym}^2 \pi \otimes \omega \times \text{Sym}^2 \pi \otimes \omega)^9 L^T(s, \text{Sym}^4 \pi \otimes \omega^2)^4 \\ &\quad \cdot L^T(s, \text{Sym}^2 \pi \otimes \omega^3)^{12} L^T(s, \omega^4)^4, \end{aligned}$$

where  $\omega$  is the central character of  $\pi$ .

From here on, we assume that  $\pi$  is self-dual and that it is not of solvable polyhedral type. We will use the results of Gelbart–Jacquet [1], Kim–Shahidi [4, 5], and Kim [3] on the automorphy of the symmetric second, third, and fourth power lifts. We will also make use of the bounds towards the Ramanujan conjecture that were obtained by Kim–Sarnak (Appendix 2 of [3]), which imply that  $|a_p| \leq 2p^{7/64}$  for all primes  $p$ . We then obtain the following results about incomplete  $L$ -functions and their associated Dirichlet series:

For  $k \leq 8$ , the incomplete  $L$ -function  $L^T(s, \pi^{\times k})$  has an absolutely convergent Euler product for  $s > 1$ . If  $k$  is even, then the incomplete  $L$ -function has a pole of order  $m(k)$  at  $s = 1$ , where  $m(k) = 1, 2, 5, 14$  for  $k = 2, 4, 6, 8$ , respectively. If  $k$  is odd, then the  $L$ -function is invertible at  $s = 1$ .

As  $s \rightarrow 1^+$ , for  $k = 2, 3$  or  $4$ , we have

$$\sum_{p \notin T} \frac{a_p^k}{p^s} = \log L^T(s, \pi^{\times k}) + O(1)$$

and for  $k = 6$  or  $8$ , we have (using positivity)

$$\sum_{p \notin T} \frac{a_p^k}{p^s} \leq \log L^T(s, \pi^{\times k}) + O(1).$$

We can then conclude:

$$\sum_{p \notin T} \frac{a_p^k}{p^s} = \begin{cases} \log(1/(s-1)) + O(1) & \text{for } k = 2 \\ O(1) & \text{for } k = 3 \\ 2 \log(1/(s-1)) + O(1) & \text{for } k = 4, \end{cases}$$

and

$$\sum_{p \notin T} \frac{a_p^k}{p^s} \leq \begin{cases} 5 \log(1/(s-1)) + O(1) & \text{for } k = 6 \\ 14 \log(1/(s-1)) + O(1) & \text{for } k = 8 \end{cases}$$

as  $s \rightarrow 1^+$ .

We also make use of the following identities:

Let  $f(x), g(x)$  be real-valued functions, and fix some point  $u \in \mathbb{R}$ . Then,

$$\begin{aligned} \lim_{x \rightarrow u^+} \sup (f(x) + g(x)) &\leq \lim_{x \rightarrow u^+} \sup f(x) + \lim_{x \rightarrow u^+} \sup g(x) \\ \lim_{x \rightarrow u^+} \sup (-f(x)) &= - \lim_{x \rightarrow u^+} \inf f(x) \end{aligned}$$

and furthermore if  $g, f$  are non-negative functions, then

$$\begin{aligned} \lim_{x \rightarrow u^+} \inf (f(x) \cdot g(x)) &\leq \lim_{x \rightarrow u^+} \inf f(x) \cdot \lim_{x \rightarrow u^+} \sup g(x) \\ \lim_{x \rightarrow u^+} \sup (f(x) \cdot g(x)) &\leq \lim_{x \rightarrow u^+} \sup f(x) \cdot \lim_{x \rightarrow u^+} \sup g(x). \end{aligned}$$

### 3. PROOF

In this subsection we will prove Theorem 1.1.

First define the sets  $A := \{p \text{ prime} \mid a_p > 0\}$  and  $B := \{p \text{ prime} \mid a_p \leq 0\}$ . From the previous section, we know that

$$\lim_{s \rightarrow 1^+} \sup \frac{\sum_p \frac{|a_p|^4}{p^s}}{\log\left(\frac{1}{s-1}\right)} = 2,$$

which implies

$$\lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^4}{p^s}}{\log\left(\frac{1}{s-1}\right)} + \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^4}{p^s}}{\log\left(\frac{1}{s-1}\right)} \geq 2.$$

Let us define  $d$

$$d := \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^4}{p^s}}{\log\left(\frac{1}{s-1}\right)},$$

and thus we can write

$$\lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^4}{p^s}}{\log\left(\frac{1}{s-1}\right)} \geq 2 - d.$$

Now define  $S_\beta \subset A$  to be exactly the set of primes  $p$  such that  $|a_p|^4 \geq (2-d)\beta$ , where  $0 < \beta < 1$  is a constant to be fixed later. We will assume that  $S_\beta$  has an upper Dirichlet density smaller than  $1/100$ .

We have the bound

$$\limsup \left( \frac{\sum_{p \in A-S_\beta} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \leq (2-d)\beta,$$

Since

$$\limsup \left( \frac{\sum_{p \in A-S_\beta} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) + \limsup \left( \frac{\sum_{p \in S_\beta} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \geq (2-d),$$

we have

$$\limsup \left( \frac{\sum_{p \in S_\beta} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \geq (2-d)(1-\beta),$$

and using Cauchy–Schwarz

$$\left( \limsup \frac{\sum_{p \in S_\beta} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^2 \leq \left( \limsup \frac{\sum_{p \in S_\beta} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \left( \limsup \frac{\sum_{p \in S_\beta} \frac{|a_p|^0}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)$$

where the third limit supremum can be bounded above by  $1/100$ , and we get

$$(3.1) \quad (2-d)^2(1-\beta)^2 \leq \frac{14}{100}.$$

We need to appeal to a result from [8] which we include here as a lemma.

**Lemma 3.1.** *For  $A, B$ , and  $d$  defined as above, we have*

$$\limsup \frac{\sum_{p \in A} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \geq \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}}.$$

*Proof.* As explained, a proof of this Lemma essentially arises in [8]. We include a proof below for the convenience of the reader.

Using Holder's inequality and taking the limit supremum as  $s \rightarrow 1^+$ ,

$$\begin{aligned} \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} &\leq \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{3/4} \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{1}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{1/4} \\ &\leq (2-d)^{3/4} \cdot 1^{1/4}. \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} &\leq \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{1/5} \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{4/5} \\ 2-d &\leq \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{1/5} (2-d)^{3/5} \\ (2-d)^2 &\leq \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)}, \end{aligned}$$

From the results in the previous section, we have

$$\lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} + \lim_{s \rightarrow 1^+} \inf \frac{\sum_{p \in B} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} \leq 14,$$

and so

$$(3.2) \quad \lim_{s \rightarrow 1^+} \inf \frac{\sum_{p \in B} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} \leq 14 - (2-d)^2.$$

We also have

$$\sum_{p \in B} \frac{|a_p|^{8/5} |a_p|^{12/5}}{p^s} \leq \left( \sum_{p \in B} \frac{|a_p|^8}{p^s} \right)^{1/5} \left( \sum_{p \in B} \frac{|a_p|^3}{p^s} \right)^{4/5}.$$

We divide the equation above by  $\log(1/(s-1))$  and take the limit infimum as  $s \rightarrow 1^+$ ,

$$\lim_{s \rightarrow 1^+} \inf \frac{\sum_{p \in B} \frac{|a_p|^4}{p^s}}{\log \left( \frac{1}{s-1} \right)} \leq \left( \lim_{s \rightarrow 1^+} \inf \frac{\sum_{p \in B} \frac{|a_p|^8}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{1/5} \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{4/5}.$$

We apply equation 3.2, to get

$$d \leq (14 - (2-d)^2)^{1/5} \left( \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^{4/5}$$

$$\frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} \leq \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)}.$$

For  $s > 1$ , we have

$$\sum_{p \in A} \frac{|a_p|^3}{p^s} + \left( - \sum_p \frac{a_p^3}{p^s} \right) = \sum_{p \in B} \frac{|a_p|^3}{p^s},$$

and so

$$\lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \geq \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)}.$$

Therefore

$$\lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in A} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \geq \lim_{s \rightarrow 1^+} \sup \frac{\sum_{p \in B} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \geq \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}}.$$

□

Now we define  $T_\alpha \subset A$  to be the set of primes  $p$  such that

$$|a_p|^3 \geq \left( \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} \right) \alpha,$$

where  $0 < \alpha < 1$  is a constant to be fixed later.

Let us assume that the upper Dirichlet density of  $T_\alpha$  is less than  $1/100$ . We have

$$\limsup \left( \frac{\sum_{p \in A-T_\alpha} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \leq \left( \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} \right) \alpha.$$

Lemma 3.1 implies that

$$\limsup \left( \frac{\sum_{p \in A-T_\alpha} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) + \limsup \left( \frac{\sum_{p \in T_\alpha} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \geq \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}},$$

then

$$\limsup \left( \frac{\sum_{p \in T_\alpha} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \geq \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} (1 - \alpha).$$

Now

$$\left( \limsup \frac{\sum_{p \in T_\alpha} \frac{|a_p|^3}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right)^2 \leq \left( \limsup \frac{\sum_{p \in T_\alpha} \frac{|a_p|^6}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right) \left( \limsup \frac{\sum_{p \in T_\alpha} \frac{|a_p|^0}{p^s}}{\log \left( \frac{1}{s-1} \right)} \right).$$

The second limit supremum can be bounded from above by 5 and the third limit supremum can be bounded from above by  $1/100$ , so

$$(3.3) \quad \left( \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} \right)^2 (1 - \alpha)^2 \leq \frac{5}{100}.$$

Now, given some value for the constant  $\beta$ , we want to fix  $\alpha$  such that

$$(3.4) \quad ((2-d)\beta)^{1/4} = \left( \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} \alpha \right)^{1/3}.$$

Given equation 3.4, if we set  $\beta = 0.495$ , we have that if  $d \leq 1.258$ , then equation 3.1 is false, contradicting the assumption that  $S_\beta$  has an upper Dirichlet density smaller than  $1/100$ , and if  $d > 1.258$ , then equation 3.3 is false, and so  $T_\alpha$  would have an upper Dirichlet density of at least  $1/100$ . Either way, since for  $\beta = 0.95$  and  $d = 1.258$  the value of equation 3.4 is  $0.36729^{1/4} = 0.778\dots$ , this implies that the set of primes

$$\{p \mid a_p(\pi) > 0.778\dots - \epsilon\}.$$

has an upper Dirichlet density at least  $1/100$ .

*Remark 1.* We determined our choice of  $\beta$  by solving the following simultaneous equations

$$\begin{aligned} (2-d)(1-\beta) &= \frac{\sqrt{14}}{10} \\ \frac{d^{5/4}}{(14 - (2-d)^2)^{1/4}} (1-\alpha) &= \frac{\sqrt{5}}{10}, \end{aligned}$$

along with equation 3.4, and we obtained  $\beta = 0.4957\dots$  and  $d = 1.2581\dots$ .

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